Adaptive Stabilization of a Mechanical System with Nonholonomic Constraints

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Abstract

The motion control problem is considered for the nonholonomic systems with unknown dynamic parameters. The proposed control algorithm is based on a method of recursive aim inequalities which allows us to solve the problem of the adaptive stabilization in the presence of uniformly bounded disturbances affecting the system. Unlike other works, the presented algorithm of the parameters estimation is given in a form of a differential equation instead of a discrete-time algorithm.

1 Introduction

This paper addresses adaptive control of mechanical systems with linear homogeneous constraints. Assume that the system under consideration is defined on the connected Riemannian n-dimensional real smooth configuration manifold M with local coordinates $q = (q_1, \dots, q_n)$. Linear homogeneous constraints are defined by a k-dimensional smooth regular codistribution $\Delta^{\perp} = \operatorname{span}\{\omega_1, \dots, \omega_k\} \subset T^*M$ and given in the local coordinates as

$$\omega_j(q)\dot{q} = 0, \; j = 1, \dots, k, \tag{1.1}$$

where ω_j are one-forms $\omega_j(q) \in T_q^*M$ which are locally smooth differential. These constraints determine an m-dimensional distribution $\Delta = \operatorname{span}\{g_1, \dots, g_m\} \subset TM$, $m = n - k$ given in every point of M as $\Delta(q) = \ker \Delta^{\perp}(q)$.

The constraints (1.1) are called holonomic if the distribution Δ is involutive, i.e. for any pair of vector fields $X, Y \in \Delta \rightarrow [X, Y] \in \Delta$, where $[\cdot, \cdot]$ is the Lie bracket. If this is the case, then by the Frobenius theorem there exist the local coordinates $s = (s_1, \dots, s_n)$, such that the constraints (1.1) can be rewritten as $\dot{\overline{q}}_i = 0$, $i = n - k + 1, \dots, n$ and one can eliminate the *dependent* coordinates $\overline{q}_i$, $i = n - k + 1, \dots, n$.

The constraints (1.1) are called nonholonomic if the distribution Δ is not involutive. In distinction to the holonomic case, the number of the coordinates can not be reduced. We shall say that the dynamic system is completely nonholonomic [1] if the dimension of involutive closure of Δ is equal to n. Note that in the holonomic case, the dimension of the involutive closure of Δ equals k. If this dimension is greater than k and less then n ("partially nonholonomic case"), then some constraints in (1.1) are integrable as above and our consideration can be reduced to the completely nonholonomic case.

A curve $q(t) : I \rightarrow M$, $I \subset R$ on the configuration manifold M is said to be *admissible* (or *compatible with the distribution* Δ), if its tangent vector at each point belongs to the corresponding distribution, i.e. $\dot{q}(t) \in \Delta$ for all $t \in I$. Locally, for any admissible curve $q(t)$, there can be found real functions $u_i(t)$, $i = 1, \dots, m$ which satisfy the following equality:

$$\dot{q} = \sum_{i=1}^{m} g_i(q) \cdot u_i. \tag{1.2}$$

This equation can be regarded as an equation that describes a control system with the input functions u^i, $i = \overline{1, m}$. We shall refer to (1.2) as a *kinematic model* of the nonholonomic motion. Applying the Chow's theorem [2], it can be shown that the completely nonholonomic control system is strongly accessible and, since the configuration manifold is connected by assumption, controllable.

The d'Alambert's principle of virtual displacements allows us to derive the dynamic equations of the unconstrained motion. A mechanical system which is subject to the linear homogeneous constraints (1.1) posses a quadratic (in the generalized velocities) Lagrangian function $L(q, \dot{q}, \theta)$ [3]. Then, the Euler-Lagrange equation takes the following form

$$M(q, \theta)\ddot{q} + C(q, \dot{q}, \theta) = \Omega'(q)\lambda + G(q)(U + \xi(t)), \tag{1.3}$$

where θ is a constant r-vector of the dynamic parameters, $\theta \in \Theta$, $\Theta \subset R^r$ is a convex set, $M(q, \theta)$ is an $(n \times n)$ symmetric positive definite inertia matrix with the property $\|M^{-1}(q,\theta)\| \leq \sigma < \infty$; $\Omega(q)$ is a $(k \times n)$-matrix with rows being the differential one-forms, constituting the nonholonomic constraints (1.1) and evaluated in q, i.e. $\Omega_{ij}(q) = \omega_i^j(q)$, $i = \overline{1,k}$, $\operatorname{rank}\Omega(q) = k$; similarly, $G(q)$ is an $(n\times m)$-matrix with columns being the vector fields, constituting the local basis of the distribution Δ and evaluated in q, i.e. $G_{ij}(q) = g_j^i(q)$, $\operatorname{rank} G(q) = m$; U is an m-vector of the external forces, $\xi(t) \in R^n$ is an external disturbance; $C(q,\dot q,\theta)$ is an n-vector of Coriolis, centripetal and gravity torques. We shall refer to (1.3) as a *dynamic model* of the nonholonomic motion. The both equations of the nonholonomic constraints (1.1) and the dynamic equation (1.3) constitute the mathematical model of the controlled motion of the nonholonomic system. It consists of $(n+k)$ equations with n generalized coordinates $q^i, i = 1,\ldots,n$ and k unknown Lagrangian multipliers $\lambda^j, j = 1,\ldots,k$.

2 Reduction of the dynamic model: the Appel's form

The Lagrange multipliers may be computed using the classical method (see, e.g. [4]). Namely, by means of differentiating the equations of the nonholonomic constraints (1.1) along (1.3) and taking into account the nonsingularuty of the matrices M and Ω, one can solve the resulting equation for λ as a function of $(q,\dot q,u)$. Then, substituting in (1.3) results in a dynamical equation of the reduced order on the constrained state space

$$\mathcal{M} = \{(q,\dot q) \in T^*M \mid \Omega(q)\dot q = 0\}. \tag{2.1}$$

A more insightful matricidial formalism of this procedure was proposed in [5, 6] for the holonomic systems and in [7] for the nonholonomic systems. It is based on an assumption that there exists a partition of the configuration variables $q = (q_1, q_2)$ (dependent and independent coordinates respectively), such that the corresponding partition of the matrix $\Omega(q) = [\Omega_1(q), \Omega_2(q)]$, where $\Omega_2(q)$ is a $(k\times k)$-matrix, satisfies the condition locally:

$$\operatorname{rank}\Omega_2(q) = k. \tag{2.2}$$

This condition is restrictive in the case, if the global stabilization is the control problem to be solved. For example, the so called *rolling and nonslipping conditions* of the nonholonomic motion on the plane has the form $\dot q_1 \sin(q_3) - \dot q_2 \cos(q_3) = 0$. One can see, that there is no *globally* nonsingular minor of the matrix $\Omega(q) = [\sin(q_3)\ \ \cos(q_3)\ \ 0]$ representing the nonholonomic constraints. Such an obstruction motivates elaboration of another method in order to eliminate the Lagrange multipliers. One can implicitly obtain the Appel's form of the dynamic equations subjected to the nonholonomic constraints (see [3] for the general consideration and [8, 9] for the nonholonomic examples). Instead, we propose a specific state transformation of the Euler-Lagrange equations (1.3) into the Appel's form.

Proposition 2.1 *Consider a partition of the generalized velocities vector $\dot q = p = [p_1, p_2]$, where $p_1 \in R^m$, $p_2 \in R^{n-m}$. Then, the Euler-Lagrange equation (1.3) for a mechanical system subjected to the linear nonholonomic constraints (1.1) is*

$$\begin{cases} \dot q = p, \\ M(q)\dot p + C(q,p) = \Omega'(q)\lambda + G(q)(U+\xi(t)), \end{cases} \tag{2.3}$$

$$\Omega(q)p = 0 \tag{2.4}$$

may be transformed via the state transformation $\mathcal{R}$: $[q,p_1,p_2] \to [q,u,\overline{p}_2]$ into the reduced Appel's dynamic equation

$$\begin{cases} \dot q = G(q)u, \\ \overline{M}(q)\dot u + \overline{C}(q,u) = \overline{G}(q)(U+\xi(t)), \end{cases} \tag{2.5}$$

which is given on the constrained space $\mathcal{M}$ in (q,u)-coordinates ($u \in R^{n-k}$ is a vector of the so called quasi-velocities*). The transformation is defined by the equation*

$$\begin{pmatrix} u \\ \overline{p}_2 \end{pmatrix} = R^{-1}(q)\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \tag{2.6}$$

where $R(q) = [G(q), \Omega'(q)]$.

In (2.5) $\overline{M}(q)$ is a $(n-k)\times(n-k)$ symmetric positive matrix, $\overline{G}(q)$ is a $(n-k)\times l$-matrix and $\overline{C}(q,u)$ is a $(n-k)$-vector given by

$$\overline{M}(q) = G'(q)M(q)G(q),\ \ \overline{G}(q) = G'(q)G(q),$$
$$\overline{C}(q) = G'(q)\,C(q,G(q)u) + M(q)\frac{\partial p}{\partial q}(q,u,\overline{p}_2 = 0)G(q)u.$$

Remark 2.2 The advantage of the approach proposed is its suitability for the holonomic, partially nonholonomic and completely nonholonomic cases. It is a “Lagrange counterpart” of the elimination algorithm presented in [10] and based on the Hamiltonian formalism.

3 Stabilizing control

Let a smooth state transformation $s = S(t,q)$ be given and bijectively maps the configuration manifold M onto R^n. The control goal is to stabilize the transformed state vector s, i.e.

$$s \to 0, \quad \text{while } t \to \infty. \tag{3.1}$$

This formulation includes stabilization to the desired trajectory as a special case.

Using the transformation $S(t,q)$, the equations (2.5) can be represented in the following form:

$$\dot{s} = f(t,s) + \tilde{G}(t,s)u, \qquad (3.2)$$

where $f(t,s) = \frac{\partial S}{\partial t}(t, S^{-1}(t,s))$, $\tilde{G}(t,s) = S_*(t,S^{-1}(t,s))\overline{G}(S^{-1}(t,s))$, and S_* is the tangent map of the state transformation $S(t,q)$.

In order to construct a control law for the dynamic system (2.5), let us suppose that the feedback which guarantees the control aim (3.1) for the kinematic subsystem is known. More precisely, we suppose that there exists the kinematic feedback law $u = \alpha(t,s)$, and a quadratic Lyapunov function $V(s) = s'Ps$, $P' = P$, $P > 0$, such that on the solutions of the closed-loop kinematic subsystem

$$\dot{s} = f(t,s) + \tilde{G}(t,s)\alpha(t,s) \qquad (3.3)$$

the inequality $\dot{V}(s) \le -\kappa V(s)$ holds for some positive κ (see [11, 12, 13] for the nonholonomic examples).

Consider an error function $e = u - \alpha(t,s)$. Using (s,e)-coordinates we can rewrite the equations (2.5) as

$$\left\{ \begin{array}{l} \dot{s} = f(t,s) + \tilde{G}(t,s)\alpha(t,s)) + \tilde{G}(t,s)e, \\ \tilde{M}(t,s,\theta)\dot{e} + \tilde{M}(t,s,\theta)D\alpha(t,s,e) \\ \quad + \tilde{C}(t,s,e,\theta) = U + \xi(t), \end{array} \right. \qquad (3.4)$$

where

$$\tilde{M}(t,s,\theta) = \overline{G}^{-1}(S^{-1}(t,s))\overline{M}(S^{-1}(t,s),\theta),$$
$$\tilde{C}(t,s,e,\theta) = \overline{G}^{-1}(S^{-1}(t,s))\overline{C}(S^{-1}(t,s), e + \alpha(t,s), \theta),$$
$$D\alpha(t,s,e) = \partial\alpha/\partial t + \partial\alpha/\partial s\, \dot{s}.$$

Theorem 3.1 *Assume that the external disturbance $\xi(t) \equiv 0$ and the vector of the dynamic parameters θ is known. Then, the control system (3.4) is globally exponentially stabilizable by the feedback*

$$\left\{ \begin{array}{l} U(t,s,e) = \beta(t,s,e,\theta) \stackrel{\text{def}}{=} \\ \tilde{M}(t,s,\theta)(D\alpha(t,s,e) - \gamma e - \tilde{G}'(t,s)Ps) \\ +\tilde{C}(t,s,e,\theta) \end{array} \right. \qquad (3.5)$$

where $\gamma > 0$ is a parameter.

4 Adaptive control

Let us consider a case of the unknown vector θ and assume that the unmeasured disturbance ξ is bounded and satisfies the inequality

$$\|\xi(t)\| \le C_\xi, \qquad (4.1)$$

where C_ξ is given. Consider a *prediction error function* $\eta(t,s,e,\dot{e},\theta,\hat{\theta}) = \tilde{M}(t,s,\hat{\theta})(\dot{e} + D\alpha(t,s,e)) + \tilde{C}(t,s,e,\hat{\theta}) - U$, where $\hat{\theta}$ is an on-line estimate of θ. It is assumed that $\hat{\theta} \in \Theta$. Then, equations (3.4) can be represented in the following form:

$$\left\{ \begin{array}{l} \dot{s} = f(t,s) + \tilde{G}(t,s)\alpha(t,s)) + \tilde{G}(t,s)e, \\ \tilde{M}(t,s,\hat{\theta})\dot{e} + \tilde{M}(t,s,\hat{\theta})D\alpha(t,s,e) + \tilde{C}(t,s,e,\hat{\theta}) \\ = U + \eta(t,s,e,\dot{e},\theta,\hat{\theta}). \end{array} \right. \qquad (4.2)$$

There are two differences between the equations (3.4) and (4.2). The latter includes the estimate $\hat{\theta}$ instead of the unknown vector θ as well as the measured error function η instead of the unmeasured disturbance $\xi(t)$. Now, we can use the equations (4.2) in order to control the system under consideration. The problem is to construct the estimate $\hat{\theta}$ which makes the prediction error function η small in some sence. This problem is solved by means of a method of the recursive aim inequalities [14]. This method usually deals with the discrete time systems. It was also applied to the continuous time systems [16] where the estimation procedure was realized as a discrete time algorithm. In distinction to the approach of [16], we describe the estimation procedure by a differential equation.

To formulate the adaptation algorithm, let us note that the function η is linear with respect to the vector $\hat{\theta}$. It is known, that the left side of the Euler-Lagrange equations (1.3) is linear for an appropriately chosen vector of the dynamical parameters θ [5]. The reduced Euler-Lagrange equations in the Appell's form (3.4) inherit this property:

$$\left\{ \begin{array}{l} \tilde{M}(t,s,\theta)(\dot{e} + D\alpha(t,s,e)) + \tilde{C}(t,s,e,\theta) \\ \quad = Y_0(t,s,e) + Y(t,s,e,\dot{e})\theta. \end{array} \right. \qquad (4.3)$$

Thus, the function η reads as $\eta(t,s,e,\dot{e},\theta,\hat{\theta}) = \xi(t) + Y(t,s,e,\dot{e})(\hat{\theta} - \theta)$. To simplify further notations, we denote $\eta(t,\hat{\theta}) = \eta(t,s,e,\dot{e},\theta,\hat{\theta})$.

Let δ and $C_\eta > C_\xi$ be some positive constants. Determine a law to update the parameters in the form of a differential equation given on a set of the admissible dynamic parameters Θ

$$\dot{\hat{\theta}} = -\delta h(\|\eta(t,\hat{\theta})\| - C_\eta)P_{\partial\Theta}(\hat{\theta})\left[Y'(t,s,e,\dot{e})\eta(t,\hat{\theta})\right], \qquad (4.4)$$

with some initial state $\hat{\theta}_0 = \hat{\theta}(t_0) \in \Theta\backslash\partial\Theta$. In (4.4) $h(\cdot)$ is a scalar function

$$h(\alpha) = \left\{ \begin{array}{ll} 0, & \alpha < 0, \\ 1, & \alpha \ge 0, \end{array} \right.$$

and $P_{\partial\Theta}(\hat{\theta})$ is an operator of the orthogonal projection of the vector field $Q(\cdot) \in TR^r$ on the boundary $\partial\Theta$ of Θ. Such an operator assigns to a vector field

$Q(\cdot, \hat{\theta})$, $\hat{\theta} \in \partial\Theta$ of the differential equation (4.4) its orthogonal projection on the tangent space to $\partial\Theta$ through a point $\hat{\theta} \in \partial\Theta$ if the vector field $Q(\cdot, \hat{\theta})$ is directed outside of the set Θ. It is supposed to be identical for all $\hat{\theta}$ belonging to the interior of Θ. Since the parameters updating law is given as the differential equation with a discontinous right side, we shall understand the solution of (4.4) in the Filippov's sence [15]. The presented algorithm of updating the parameters involves the *dead zone* which freezes the parameters in the case if the error function satisfies the inequality $||\eta(t, \hat{\theta}(t))|| \le C_\eta$.

Theorem 4.1 *Let the external disturbance $\xi(t)$ be continuous and satisfy the inequality (4.1). Then, for any $\theta \in \Theta$ and any continuous control input $U(t)$ the prediction error function $\eta(t, \hat{\theta})$ satisfies the inequality*

$$\int_0^T h(||\eta(t, \hat{\theta})|| - C_\eta)\ ||\eta(t, \hat{\theta})||\ dt \ \le \ \frac{||\theta - \hat{\theta}(t_0)||^2}{2\delta(C_\eta - C_\xi)} \tag{4.5}$$

on the solution of the system (3.4), (4.4) for an arbitrary $T \le \infty$ that belongs to the interval where the solution of this system exists.

To complete the formulation of the adaptive control algorithm, we define the control law by replacing the unknown vector θ in (3.5) by its estimate $\hat{\theta}$:

$$U = \beta(t, s, e, \hat{\theta}). \tag{4.6}$$

Theorem 4.2 (Adaptive stabilization) *For any $E > 0$ and $\epsilon > 0$, there exist parameters γ and δ, such that any solution of the closed–loop system (3.4), (4.4), (4.6) with an initial state satisfying $|s(0)| + |e(0)| \le E$ satisfies the inequality*

$$|s(t)| + |e(t)| \le \epsilon \tag{4.7}$$

for all sufficiently large t.

5 Conclusion

The adaptive control algorithm is proposed for the reference trajectory stabilization of nonholonomic systems. It has been proved that the proposed control scheme ensures the motion stabilization with any given precision.

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